Fuzzy Sets and Fuzzy Logic

Fuzzy if-then rules

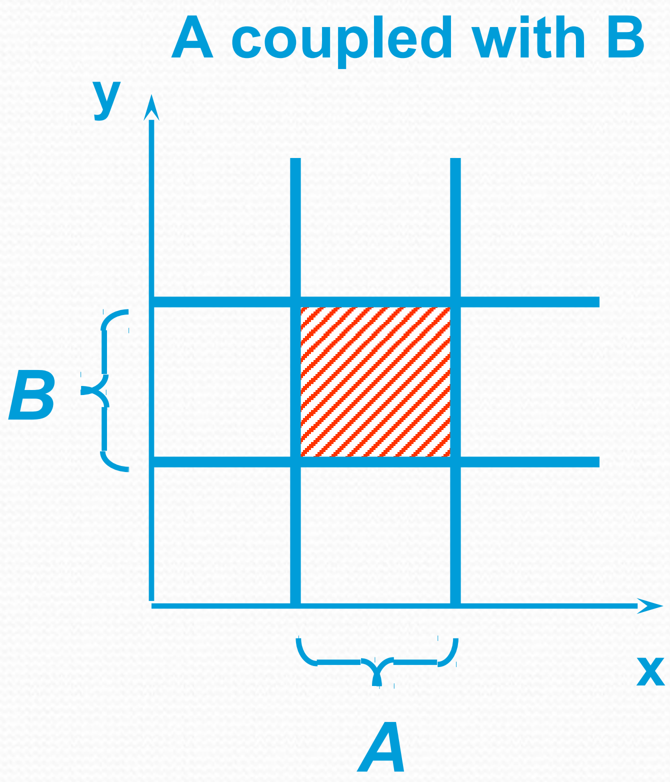
- General format:

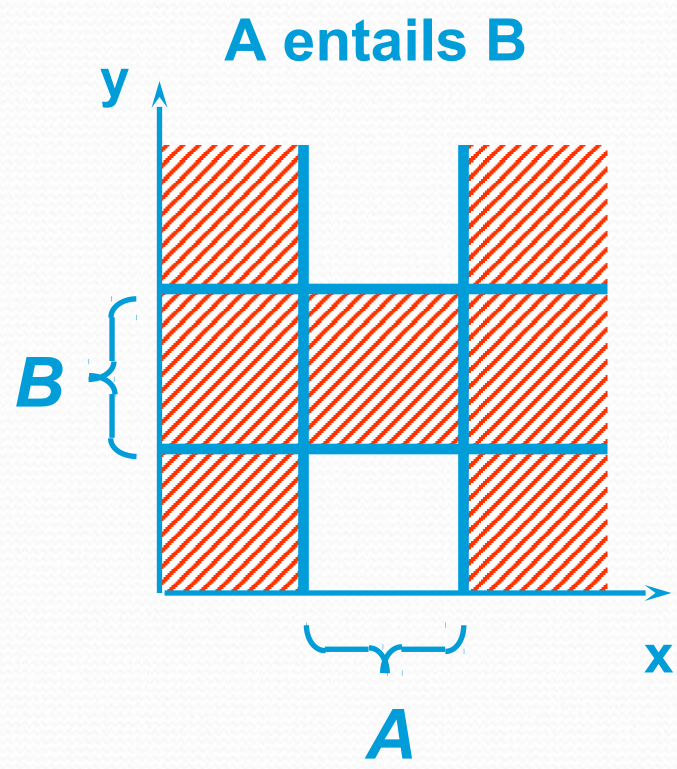
  If x is A then y is B

- Examples:
  - If pressure is high, then volume is small
  - If a restaurant is expensive, then order small dishes
  - If a tomato is red, then it is ripe
  - If the speed is high, then apply the brake a little

Interpretation

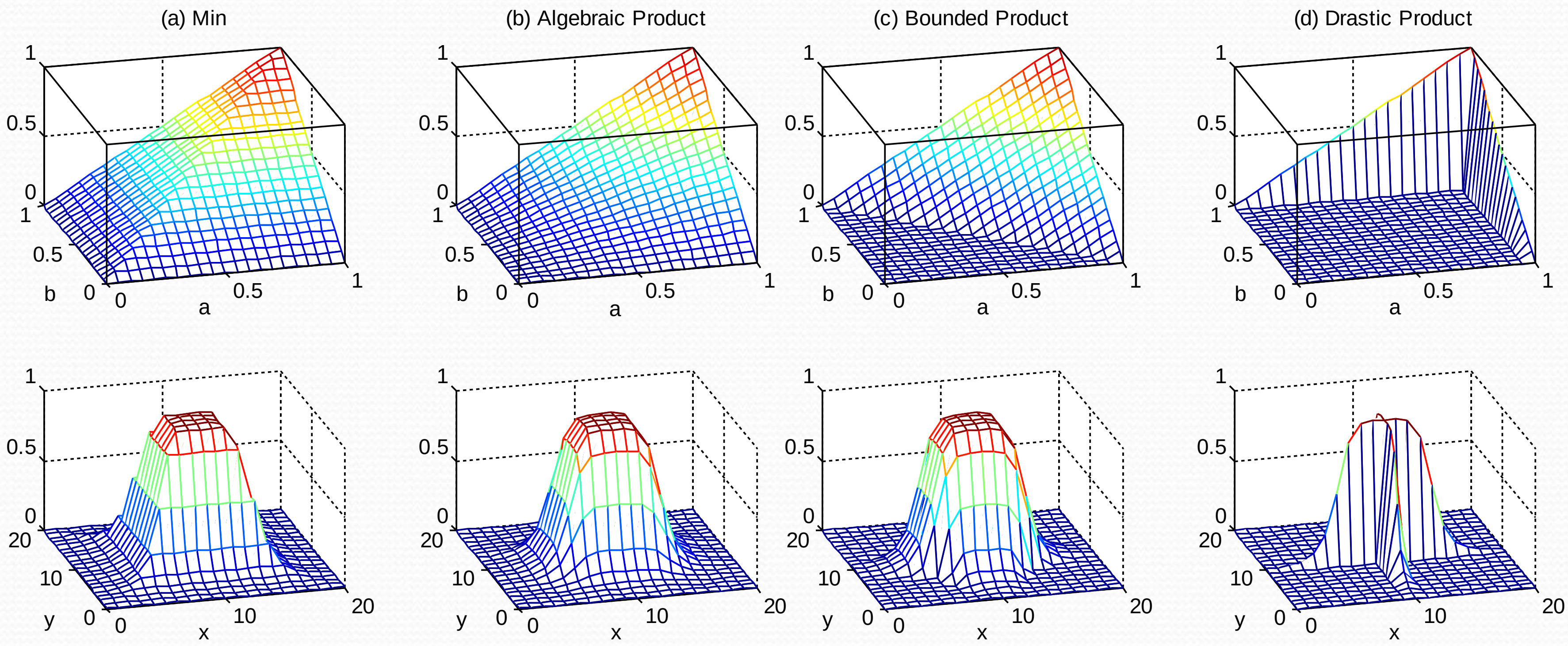
Implication in traditional logic





A coupled with B

Use the T-norm...



A entails B

- Boolean fuzzy implication (based on $\neg A \vee B$)

$$m_R(x, y) = max(1 - m_A(x), m_B(y))$$

- Zadeh's max-min implication (based on $\neg A \vee (A \wedge B)$)

$$m_R(x, y) = max(1 - m_A(x), min(m_A(x), m_B(y)))$$

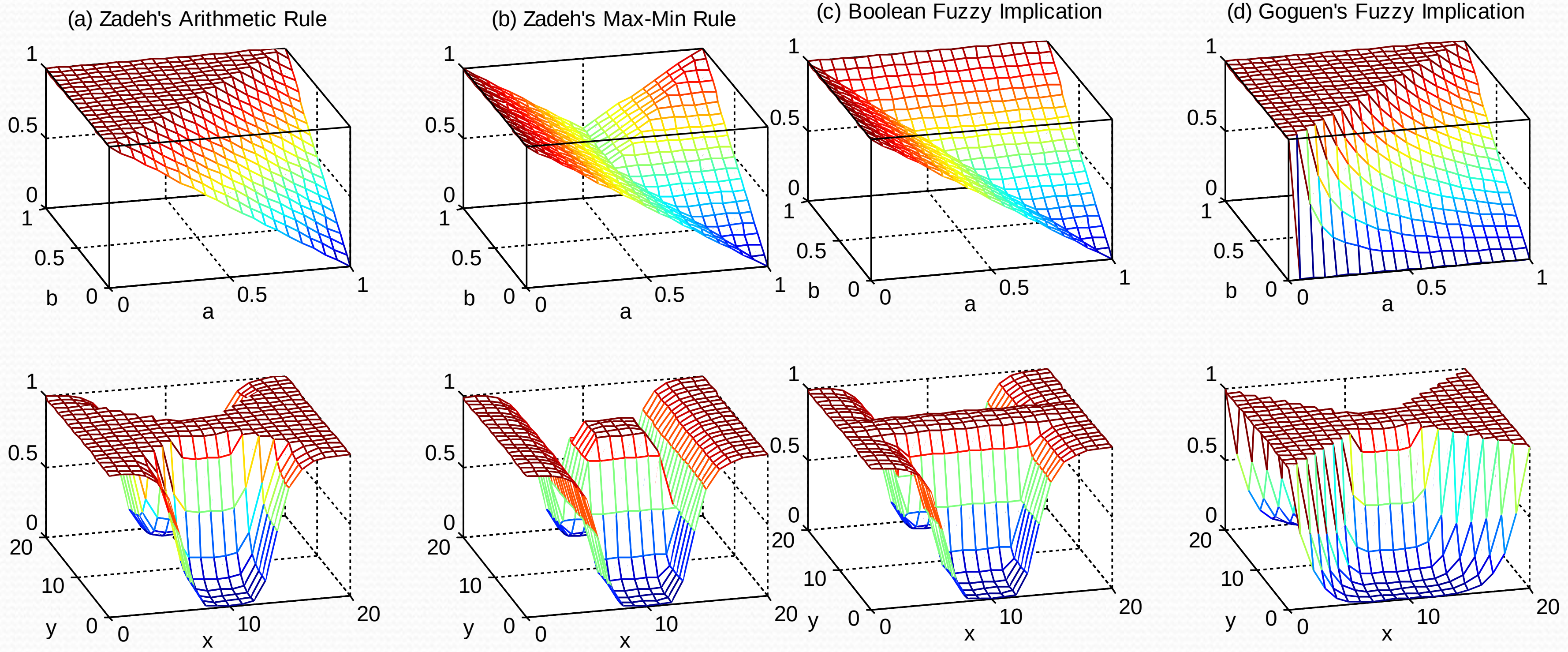
- Zadeh's arithmetic implication (based on $\neg A \vee B$)

$$m_R(x, y) = min(1 - m_A(x) + m_B(y), 1)$$

- Goguen's implication

$$m_R(x, y) = min(m_B(x) / m_A(y), 1)$$

A entails B



(a) Zadeh's Arithmetic Rule
(b) Zadeh's Max-Min Rule
(c) Boolean Fuzzy Implication
(d) Goguen's Fuzzy Implication
a
b
x
y

Fuzzy relation

A fuzzy relation R between X and Y is a 2-D fuzzy subset of X x Y

$$R = \{((x, y), \mu_R(x, y)) | (x, y) \in X \times Y\}$$

with

$$\mu_R : X \times Y \rightarrow [0,1]$$

Examples:

- x is close to y
- x and y are similar
- x and y are related (dependent)

Relations are essentially predicates with multiple arguments.

Discrete fuzzy relations

Relation: “is an important trade partner of”

	Holland	Germany	USA	Japan
Holland	1	0.9	0.5	0.2
Germany	0.3	1	0.4	0.2
USA	0.3	0.4	1	0.7
Japan	0.6	0.8	0.9	1

Constructing fuzzy relations

Premise: Young people make long GSM calls

X = {18, 20, 22, 25, 30} [years]

Y = {1, 3, 5, 7, 10, 20} [min./call]

young(x)

x	18	20	22	25	30
$\mu(x)$	1	1	0.8	0.5	0.2

long(y)

y	1	3	5	7	10	20
$\mu(y)$	0	0.1	0.2	0.5	0.9	1

Constructing fuzzy relations

Compute cylindrical extensions

young(x) into X x Y

x	$\mu(x)$	y: 1	3	5	7	10	20
18	1	1	1	1	1	1	1
20	1	1	1	1	1	1	1
22	0.8	0.8	0.8	0.8	0.8	0.8	0.8
25	0.5	0.5	0.5	0.5	0.5	0.5	0.5
30	0.2	0.2	0.2	0.2	0.2	0.2	0.2

long(y) into X x Y

y	1	3	5	7	10	20
$\mu(y)$	0	0.1	0.2	0.5	0.9	1
	0	0.1	0.2	0.5	0.9	1
	0	0.1	0.2	0.5	0.9	1
	0	0.1	0.2	0.5	0.9	1
	0	0.1	0.2	0.5	0.9	1
	0	0.1	0.2	0.5	0.9	1

Constructing fuzzy relations

Compute the aggregation of the two cylindrical extensions. Since "young" and "long" go together, you can use a conjunctive operator (e.g. minimum).

1	1	1	1	1	1
1	1	1	1	1	1
0.8	0.8	0.8	0.8	0.8	0.8
0.5	0.5	0.5	0.5	0.5	0.5
0.2	0.2	0.2	0.2	0.2	0.2

0	0.1	0.2	0.5	0.9	1
0	0.1	0.2	0.5	0.9	1
0	0.1	0.2	0.5	0.9	1
0	0.1	0.2	0.5	0.9	1
0	0.1	0.2	0.5	0.9	1

min →

x \ y	1	3	5	7	10	20
18	0	0.1	0.2	0.5	0.9	1
20	0	0.1	0.2	0.5	0.9	1
22	0	0.1	0.2	0.5	0.8	0.8
25	0	0.1	0.2	0.5	0.5	0.5
30	0	0.1	0.2	0.2	0.2	0.2

R

Projection

x	y / $\mu(x) \setminus \mu(y)$	1 / 0	3 / 0.1	5 / 0.2	7 / 0.5	10 / 0.9	20 / 1	$\Pr_X(R)$
18	1	0	0.1	0.2	0.5	0.9	1	1
20	1	0	0.1	0.2	0.5	0.9	1	1
22	0.8	0	0.1	0.2	0.5	0.8	0.8	0.8
25	0.5	0	0.1	0.2	0.5	0.5	0.5	0.5
30	0.2	0	0.1	0.2	0.2	0.2	0.2	0.2
	$\Pr_Y(R)$	0	0.1	0.2	0.5	0.9	1	

young

long

Max-min composition

The max-min composition of two fuzzy relations R (defined on X and Y) and S (defined on Y and Z) is

$$\mu_{R \circ S}(x,z) = \bigvee_y [\mu_R(x,y) \wedge \mu_S(y,z)]$$

The result is the combined relation defined on X and Z

$$R \circ S(x,z) \leftrightarrow \exists y R(x,y) \wedge S(y,z)$$

Max-min composition

example

R **S** **R∘S**

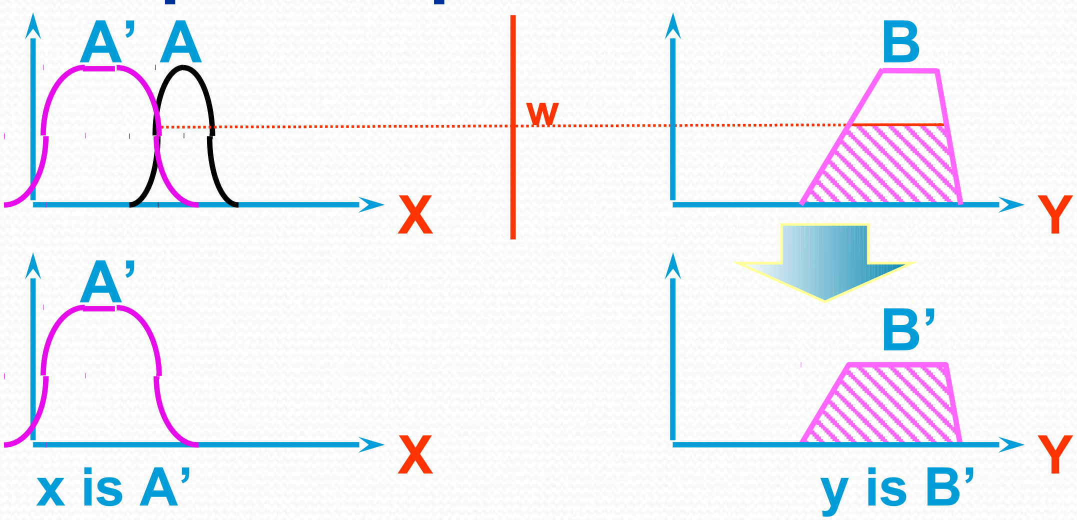
$$\begin{bmatrix} 0 & 0.1 & 0.2 & 0.5 & 0.9 & 1 \\ 0 & 0.1 & 0.2 & 0.5 & 0.9 & 1 \\ 0 & 0.1 & 0.2 & 0.5 & 0.8 & 0.8 \\ 0 & 0.1 & 0.2 & 0.5 & 0.5 & 0.5 \\ 0 & 0.1 & 0.2 & 0.2 & 0.2 & 0.2 \end{bmatrix} \circ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0.1 & 0.1 \\ 0 & 0.2 & 0.2 \\ 0 & 0.5 & 0.5 \\ 0 & 0.5 & 0.9 \\ 0 & 0.5 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0.5 & 1 \\ 0 & 0.5 & 1 \\ 0 & 0.5 & 0.8 \\ 0 & 0.5 & 0.5 \\ 0 & 0.2 & 0.2 \end{bmatrix}$$

Single rule, single antecedent

- Rule: if x is A then y is B
- Fact: x is A′
- Conclusion: y is B′

$$\mu_{B'}(y) = [\vee_x (\mu_{A'}(x) \wedge \mu_A(x))] \wedge \mu_B(y)$$
$$= w \wedge \mu_B(y)$$

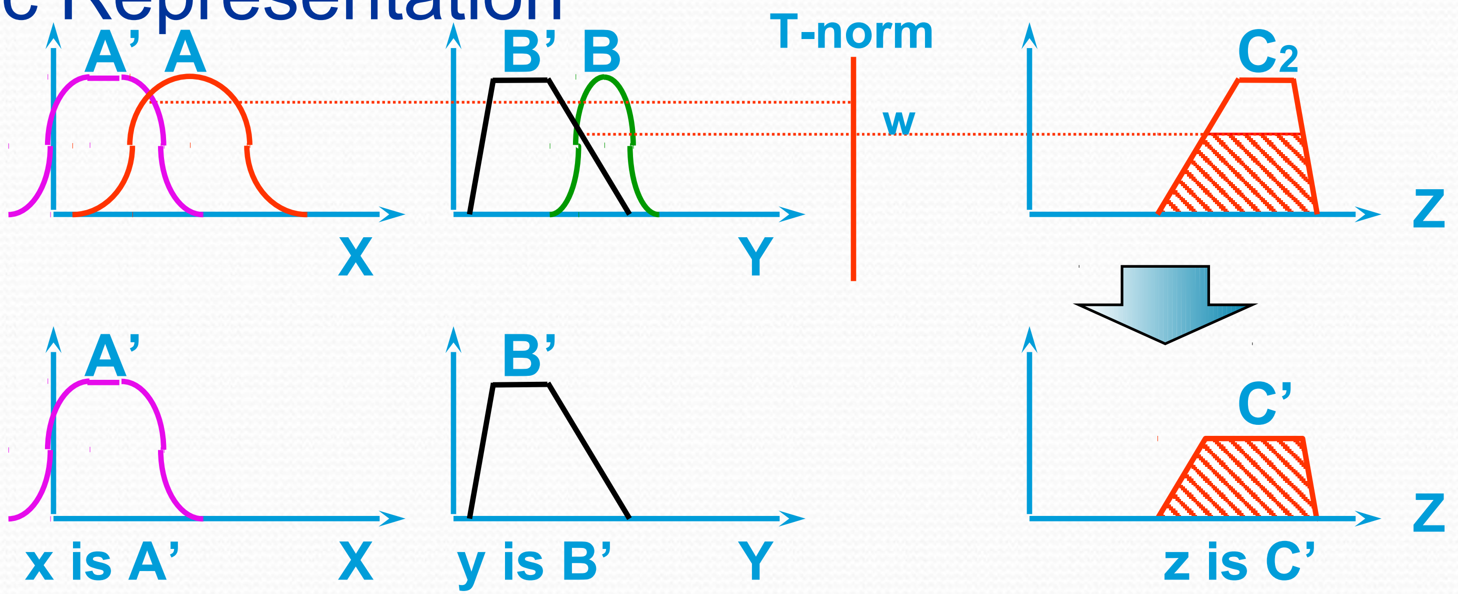
Graphic Representation



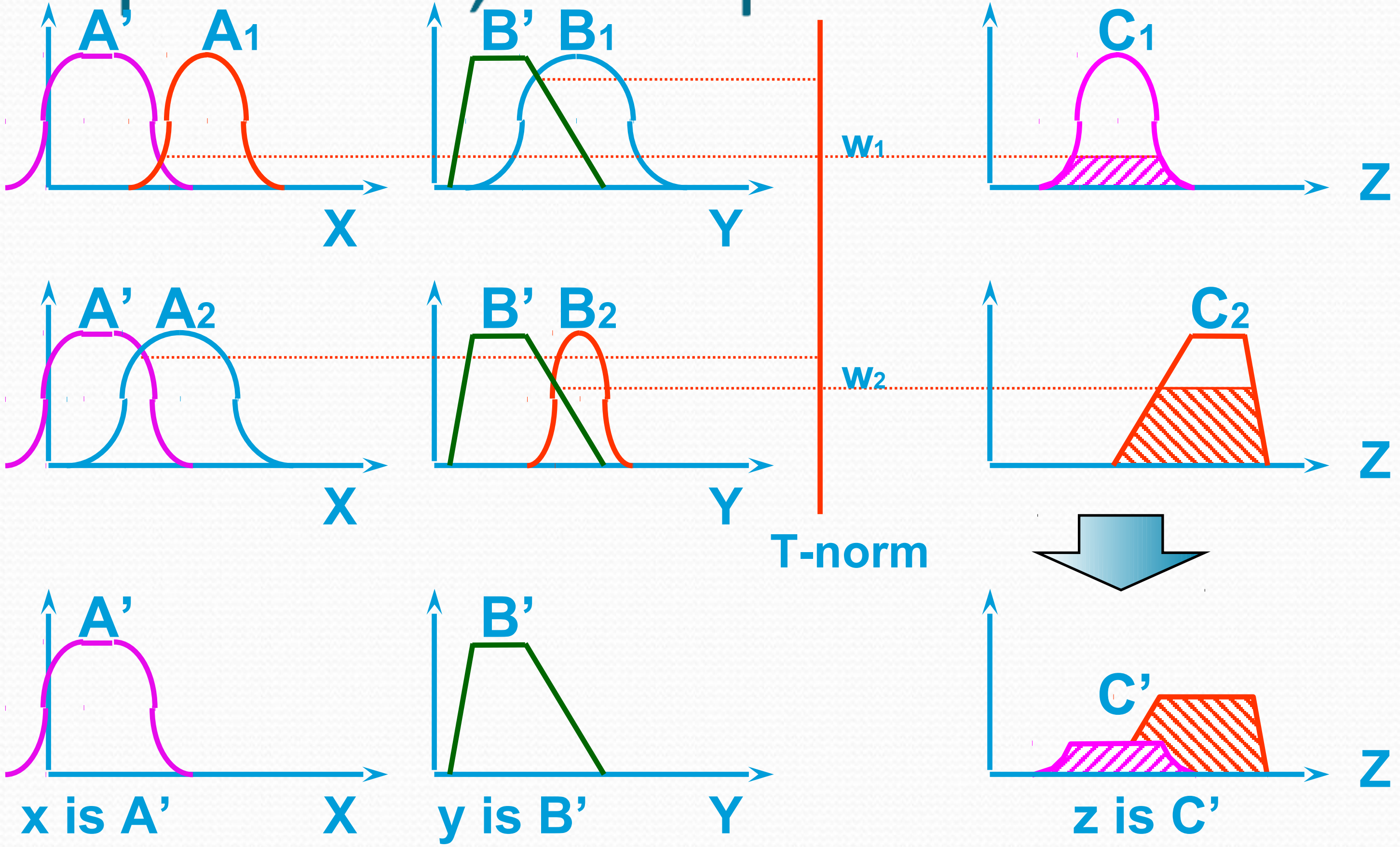
Single rule, multiple antecedents

- Rule: if x is A and y is B then z is C
- Fact: x is A′ and y is B′
- Conclusion: z is C′

Graphic Representation



Multiple rules, multiple antecedents



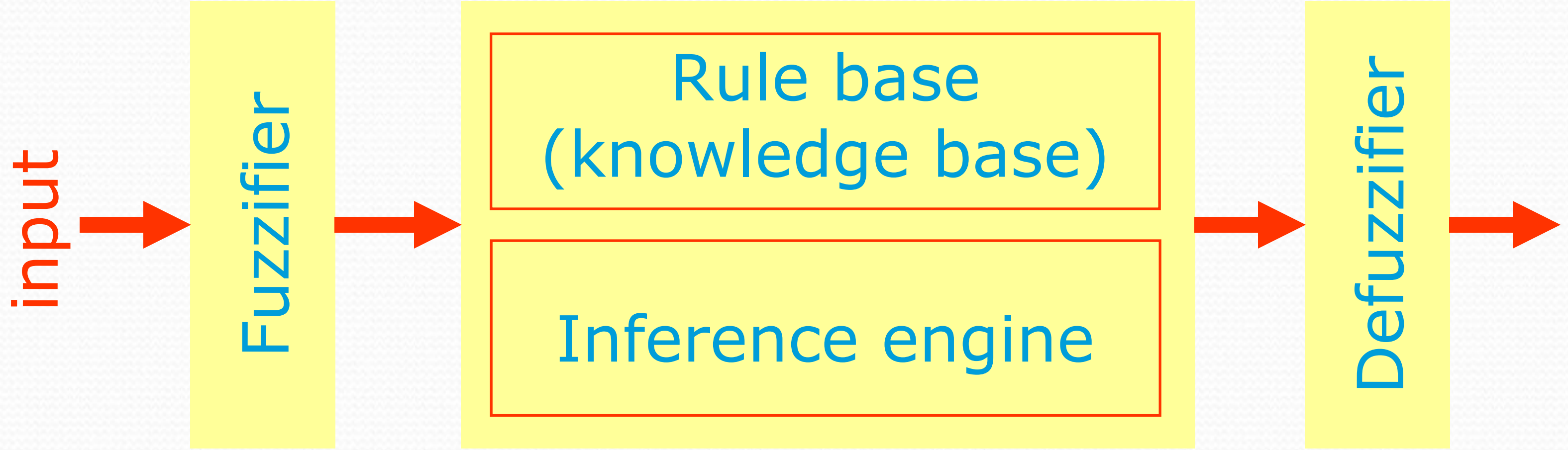
Fuzzy inference system

Multiple names

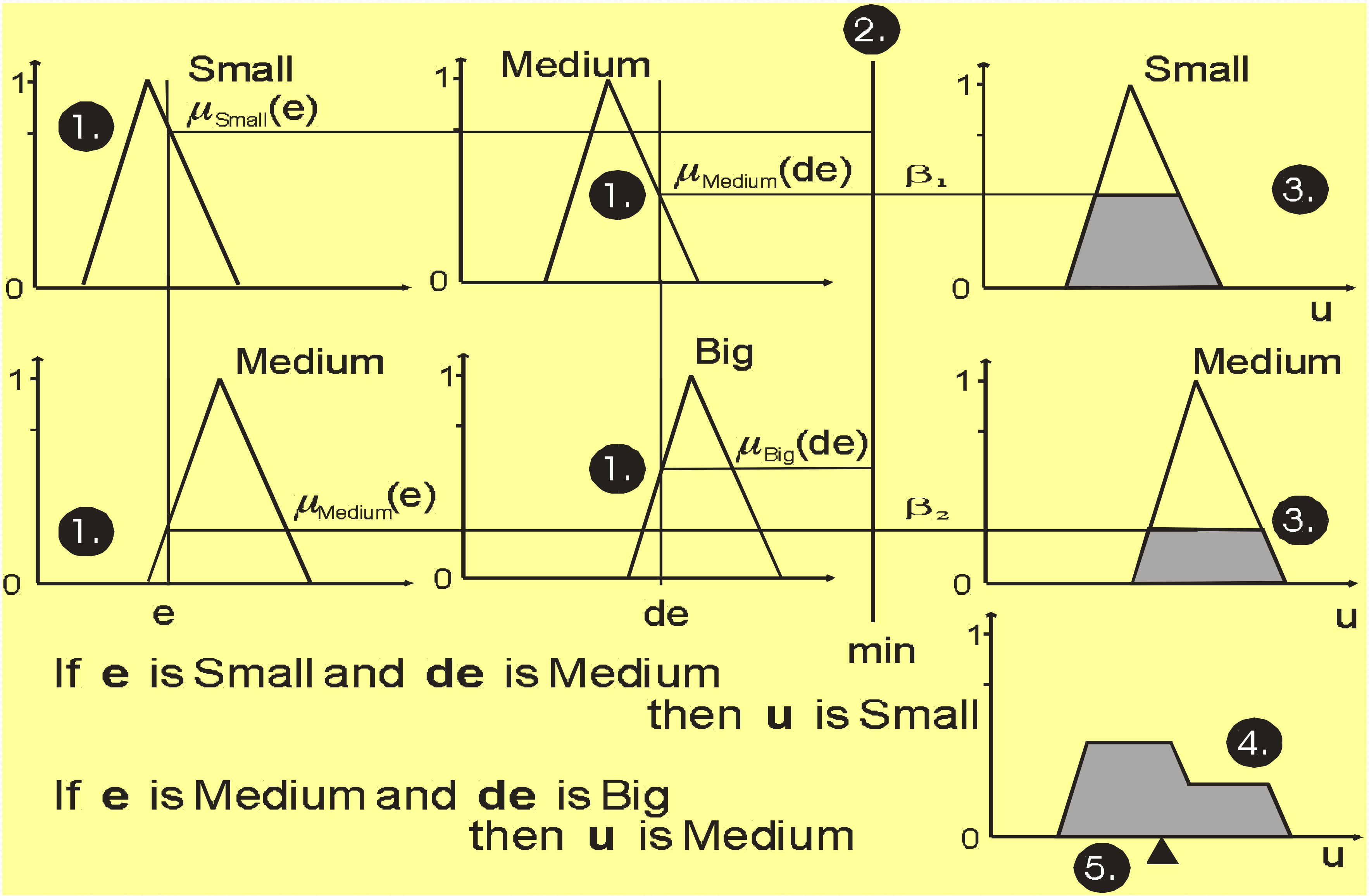
- Fuzzy rule-based system
- Fuzzy expert system
- Fuzzy model
- Fuzzy associative memory
- Fuzzy logic controller
- Fuzzy system (simply)

Building blocks

- Fuzzifier (in the simplest case, turn a measurement in a crisp set)
- Rule base
- Inference engine
- Defuzzifier



Mamdani system - example



If **e** is Small and **de** is Medium then **u** is Small

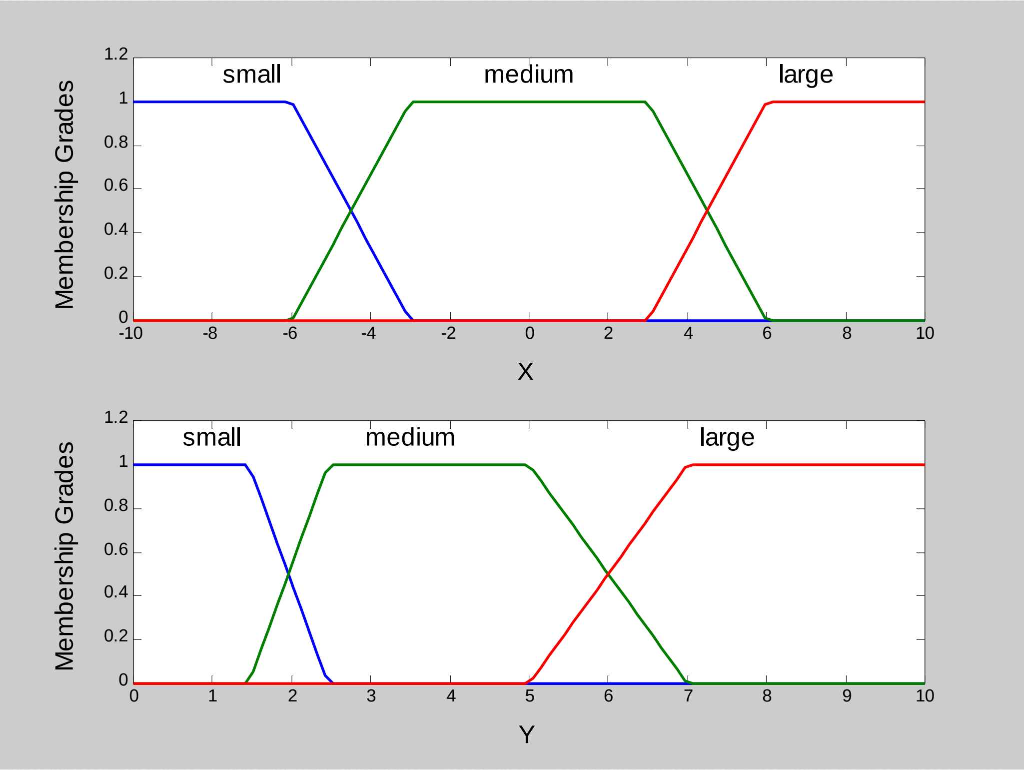
If **e** is Medium and **de** is Big then **u** is Medium

Defuzzification rules

- Centroid-of-area $$z^* = \frac{\int_Z \mu_A(z)\, z\, dz}{\int_Z \mu_A(z)\, dz}$$
- Bisector of area $$\int_{-\infty}^{z^*} \mu_A(z)dz = \int_{z^*}^{\infty} \mu_A(z)dz$$
- Mean of maximum $$z^* = \frac{\int_{Z'} z\, dz}{\int_{Z'} dz}, \quad Z' = \{z \mid \mu_A(z) = \mu^*\}$$
- Smallest of maximum $$\min_{z \in Z'} z$$
- Largest of maximum $$\max_{z \in Z'} z$$

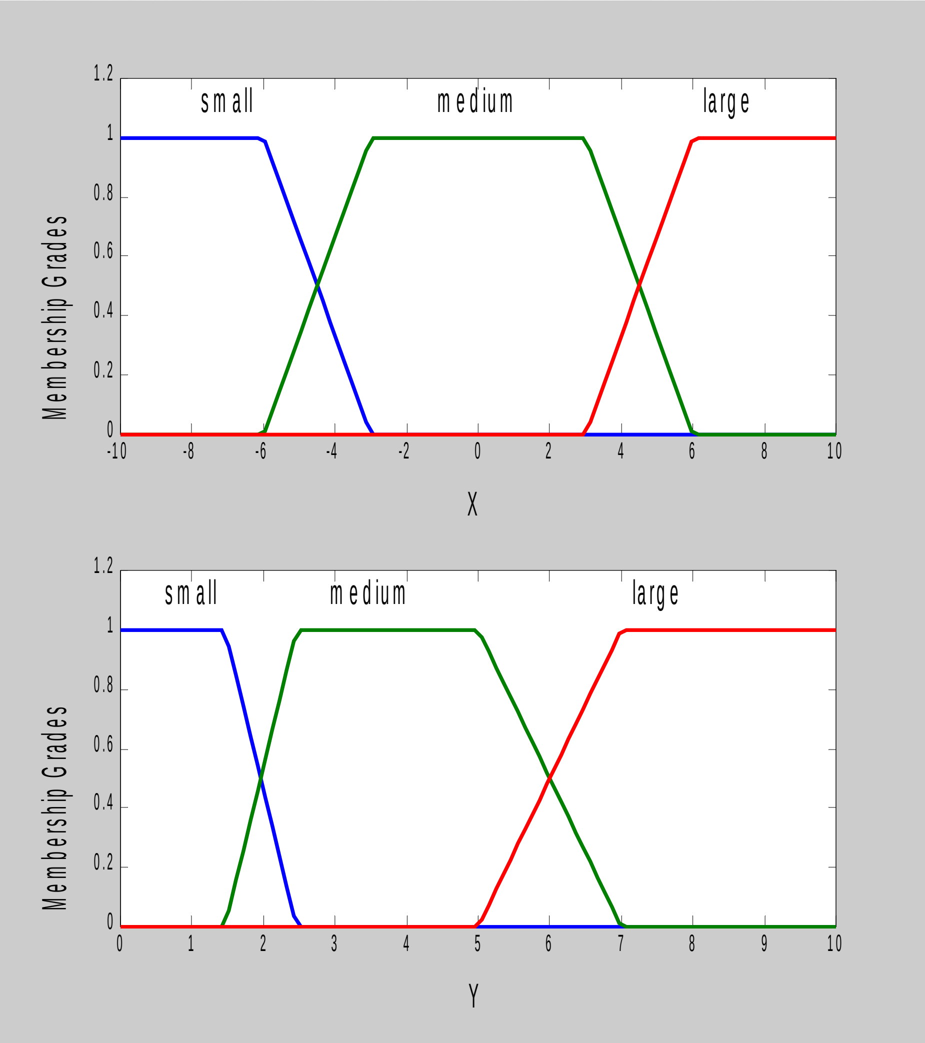
range of values where membership is maximal

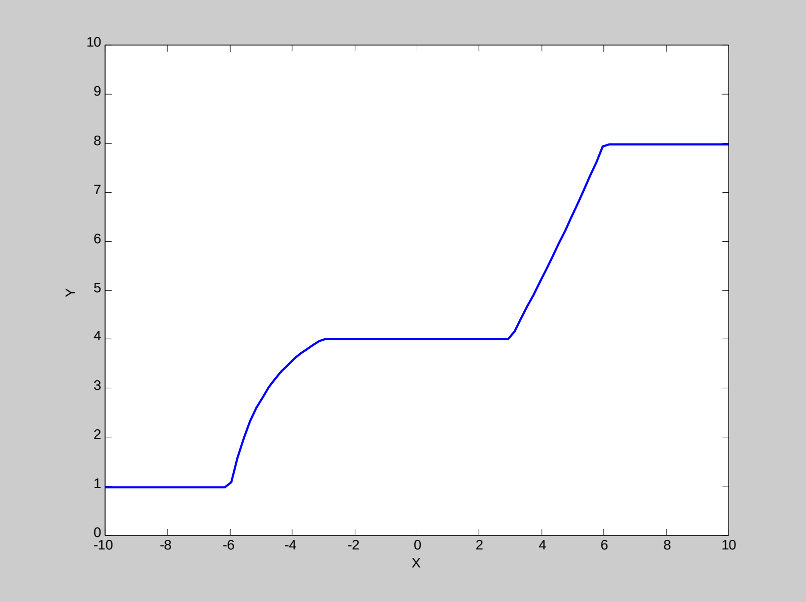
Mamdani - single input



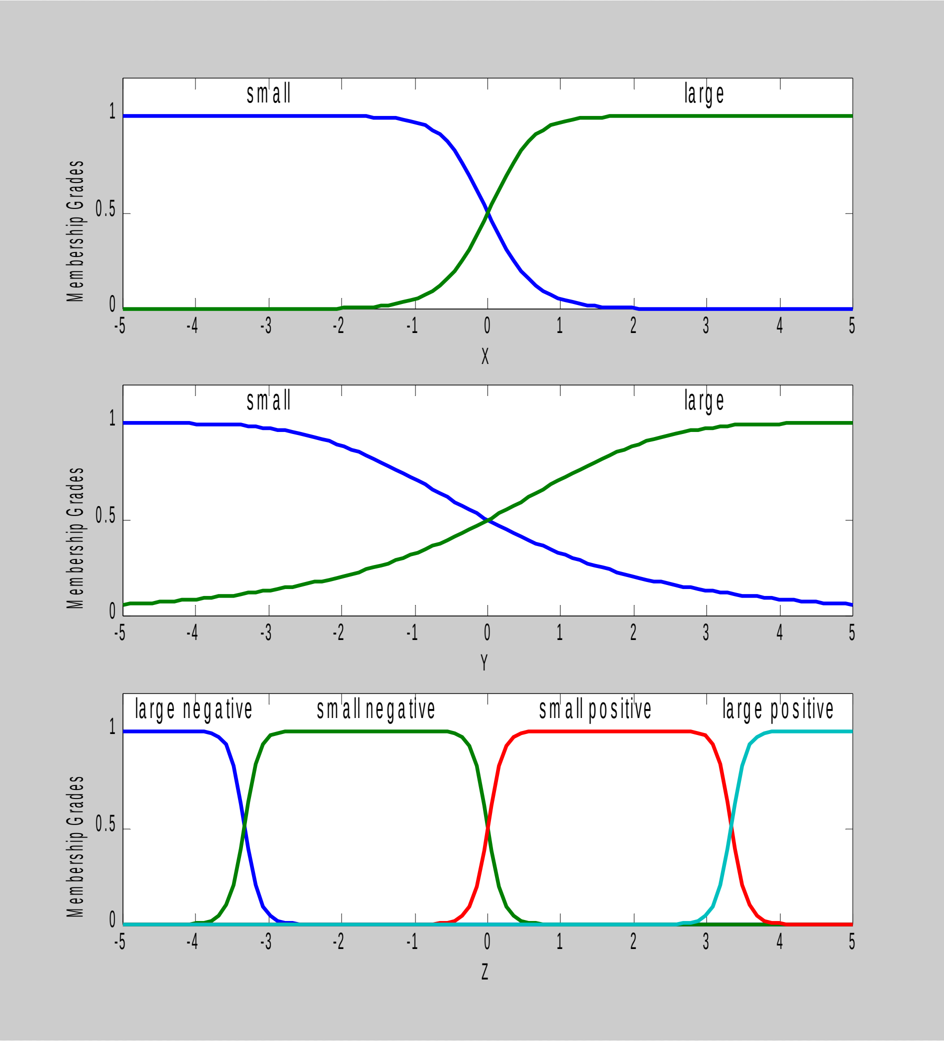
- X is Small → Y is Small
- X is Medium → Y is Medium
- X is Large → Y is Large

Mamdani - single input





Mamdani - double input



- X is Small and Y is Small → Z is negative Large
- X is Small and Y is Large → Z is negative Small
- X is Large and Y is Small → Z is positive Small
- if X is Large and Y is Large → Z is positive Large

Mamdani - double input